

## Quasiparticle universes in Bose-Einstein condensates

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Recent developments in simulating fundamental quantum field theoretical effects in the kinematical context of analogue gravity are reviewed. Specifically, it is argued that a curved space-time generalization of the Unruh-Davies effect – the Gibbons-Hawking effect in the de Sitter space-time of inflationary cosmological models – can be implemented and verified in an ultracold gas of bosonic atoms.

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### 1. The concept of an effective space-time metric

Curved space-times are familiar from Einstein's theory of gravitation,<sup>1</sup> where the metric tensor  $g_{\mu\nu}$ , describing distances in a curved space-time with local Lorentz invariance, is determined by the solution of the Einstein equations. A major problem for an experimental investigation of the (kinematical as well as dynamical) properties of curved space-times is that generating a significant curvature, equivalent to a (relatively) small curvature radius, is a close to impossible undertaking in manmade laboratories. For example, the effect of the gravitation of the whole Earth is to cause a deviation from flat space-time on this planet's surface of only the order of  $10^{-8}$  (the ratio of Schwarzschild and Earth radii). The fact that proper gravitational effects are intrinsically small is basically due to the smallness of Newton's gravitational constant  $G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ sec}^{-2}$ . Various fundamental classical and quantum effects in strong gravitational fields, a few of which we discuss below, are thus inaccessible for Earth-based experiments. The realm of strong gravitational fields (or, equivalently, rapidly accelerating a reference frame to simulate gravity according to the equivalence principle), is therefore difficult to reach. However, Earth-based experiments are desirable, because they have the obvious advantage that they can be prepared and, in particular, repeated under possibly different conditions at will.

The formalism to be described in what follows is aimed at the realization of *effective* curved space-time geometries in perfect fluids, which can indeed be prepared on Earth, and which mimic the effects of gravity inasmuch the kinematical properties

of curved space-times are concerned. Among such perfect fluids are Bose-Einstein condensates, i.e., the dilute matter-wave-coherent gases formed if cooled to ultralow temperatures, where the critical temperatures are of order  $T_c \sim 100 \text{ nK} \cdots 1 \mu\text{K}$ ; for a short review of the (relatively) recent status of this rapidly developing field see Ref. <sup>2</sup>. In what follows, it will be of some importance that Bose-Einstein condensates belong to a special class of quantum perfect fluids, so-called superfluids.<sup>3</sup>

The curved space-times we have in mind are experienced by sound waves propagating on the background of a spatially and temporally inhomogeneous perfect fluid. Of primary importance is, first of all, to realize that the identification of sound waves propagating on an inhomogeneous background, which is itself determined by a solution of Euler and continuity equations, and photons propagating in a curved space-time, which is determined by a solution of the Einstein equations, is of a *kinematical* nature. That is, the space-time metric is fixed externally by a background field obeying the laws of hydrodynamics (which is prepared by the experimentalist), and not self-consistently by a solution of the Einstein equations  $G_{\mu\nu} = 8\pi T_{\mu\nu}$  (where  $G$  and the speed of light are set to unity). The latter equations relate space-time curvature – represented by the Einstein tensor  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ , where  $R_{\mu\nu}$  is the Ricci tensor and  $R = g^{\mu\nu}R_{\mu\nu}$  the Ricci scalar – with the energy-momentum content of all other fundamental quantum fields. This energy-momentum content is represented by the classical quantity  $T_{\mu\nu}$ , which is the regularized expectation value of a quantum energy-momentum tensor.<sup>4</sup>

As a first introductory step to understand the nature of the proposed analogy, consider the wave equation for the velocity potential of the sound field  $\Phi$ , which in a homogeneous medium at rest reads

$$\left[ -\frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} + \Delta \right] \Phi = 0, \quad (1)$$

where  $c_s$  is the sound speed. It is a constant in space and time for such a medium. This equation has Lorentz invariance, that is, if we replace the speed of light by the speed of sound, it retains the form shown above in the new space-time co-ordinates obtained after Lorentz-transforming to a frame moving at a constant speed less than the sound speed. Just as the light field *in vacuo* is a proper relativistic field, sound is a “relativistic” field.<sup>a</sup> The Lorentz invariance can be made more manifest by writing equation (1) in the form  $\square\Phi \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = 0$ , where  $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  is the (contravariant) flat space-time metric (we choose throughout the signature of the metric as specified here), determining the fundamental light-cone structure of Minkowski space;<sup>5</sup> we employ the summation over equal greek indices  $\mu, \nu, \dots$ . Assuming, then, the sound speed  $c_s = c_s(\mathbf{x}, t)$  to be local in space and time, and employing the curved space-time version of the 3+1D Laplacian  $\square$ ,<sup>1</sup> one can write down the sound wave equation in an *inhomogeneous medium* in the generally co-

<sup>a</sup>More properly, we should term this form of Lorentz invariance *pseudorelativistic* invariance. We will however use for simplicity “relativistic” as a generic term if no confusion can arise therefrom.

variant form<sup>6,7</sup>

$$\frac{1}{\sqrt{-\mathbf{g}}} \partial_\mu (\sqrt{-\mathbf{g}} \mathbf{g}^{\mu\nu} \partial_\nu \Phi) = 0. \quad (2)$$

Here,  $\mathbf{g} = \det[\mathbf{g}_{\mu\nu}]$  is the determinant of the (covariant) metric tensor. It is to be emphasized at this point that because the space and time derivatives  $\partial_\mu$  are covariantly transforming objects in (2), the primary object in the condensed-matter identification of space-time metrics via the wave equation (2) is the contravariant metric tensor  $\mathbf{g}^{\mu\nu}$ .<sup>24</sup> In the condensed-matter understanding of analogue gravity, the quantities  $\mathbf{g}^{\mu\nu}$  are *material-dependent* coefficients. They occur in a dispersion relation of the form  $\mathbf{g}^{\mu\nu} k_\mu k_\nu = 0$ , where  $k_\mu = (\omega/c_s, \mathbf{k})$  is the covariant wave vector, with  $\hbar \mathbf{k}$  the ordinary spatial momentum (or quasi-momentum in a crystal).

The contravariant tensor components  $\mathbf{g}^{\mu\nu}$ , for a perfect, irrotational liquid turn out to be<sup>6,7,8</sup>

$$\mathbf{g}^{\mu\nu} = \frac{1}{A_c c_s^2} \begin{pmatrix} -1 & -\mathbf{v} \\ -\mathbf{v} & c_s^2 \mathbf{1} - \mathbf{v} \otimes \mathbf{v} \end{pmatrix}, \quad (3)$$

where  $\mathbf{1}$  is the unit matrix and  $A_c$  a space and time dependent function, to be determined from the proper equations of motion for the sound field (see below). Inverting this expression according to  $\mathbf{g}^{\beta\nu} \mathbf{g}_{\nu\alpha} = \delta^\beta_\alpha$ , to obtain the covariant metric  $\mathbf{g}_{\mu\nu}$ , the fundamental tensor of distance reads

$$\mathbf{g}_{\mu\nu} = A_c \begin{pmatrix} -(c_s^2 - \mathbf{v}^2) & -\mathbf{v} \\ -\mathbf{v} & \mathbf{1} \end{pmatrix}, \quad (4)$$

where the line element is  $ds^2 = \mathbf{g}_{\mu\nu} dx^\mu dx^\nu$ . This form of the metric has been derived by Unruh for an irrotational perfect fluid described by Euler and continuity equations;<sup>6</sup> its properties were later on explored in more detail in particular by Visser.<sup>7</sup> I also mention that an earlier derivation of Unruh's form of the metric exists, from a somewhat different perspective; it was performed by Trautman.<sup>8</sup>

The conformal factor  $A_c$  in (4) depends on the spatial dimension of the fluid. It may be unambiguously determined by considering the action of the velocity potential fluctuations above an inhomogeneous background, identifying this action with the action of a minimally coupled scalar field in  $D + 1$ -dimensional space-time:<sup>9,10</sup>

$$\begin{aligned} S &= \int d^{D+1}x \frac{1}{2g} \left[ - \left( \frac{\partial}{\partial t} \Phi - \mathbf{v} \cdot \nabla \Phi \right)^2 + c_s^2 (\nabla \Phi)^2 \right] \\ &\equiv \frac{1}{2} \int d^{D+1}x \sqrt{-\mathbf{g}} \mathbf{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi, \end{aligned} \quad (5)$$

where it is assumed that the compressibility  $1/g$  of the (barotropic) fluid,  $g = d(\ln m\rho)/dp$ , where  $p$  is the pressure and  $\rho$  the density of the fluid, is a constant. Using the above canonical identification, it may easily be shown that the conformal factor is given by  $A_c = (c_s/g)^{4-D}$ . It is mentioned here that the case of one spatial dimension ( $D = 1$ ) is special, in that the so-called conformal invariance in two space-time dimensions implies that the classical equations of motion are invariant

(take the same form) for any space and time dependent choice of the conformal factor  $A_c$ .

The line element  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$  gives us the distances travelled by the phonons in an effective space-time world in which the scalar field  $\Phi$  “lives”. In particular, quasiclassical (large momentum) phonons will follow *light*-like, that is, here, *sound*-like geodesics in that space-time, according to  $ds^2 = 0$ . Particularly noteworthy is the simple fact that the constant time slices obtained by setting  $dt = 0$  in the line element are conformally flat, i.e. the quasiparticle world looks on constant time slices like the ordinary (Newtonian) lab space, with a simple Euclidean metric in the case of Cartesian spatial co-ordinates we display.<sup>b</sup> All the intrinsic curvature of the effective space-time is therefore encoded in the metric tensor elements  $g_{00}$  and  $g_{0i}$ . Together with Matt Visser, I described this curvature and its properties for an isotachic fluid (i.e., having a speed of sound independent of space and time).<sup>12</sup> Using that phonons move on geodesics, we discovered in<sup>13</sup> the phenomenon that a vortex acts on quasiclassical phonons as an effective gravitational lens. In Ref.<sup>14</sup>, using the fact that there is curvature in any spatially inhomogeneous flow (that is, a flow which is not a simple superposition of translational motion and rigid body rotation), we have shown that there exists a sonic analogue of the “warp-drive” in general relativity permitting superluminal, i.e., here “superphononic” motion.<sup>15</sup> The point made by us is that in the acoustic curved space-times we consider, there is no violation of any condition on the positivity of energy necessary, which is in marked contrast to the original warp drives, where local energy densities by necessity must be negative to permit superluminal travel.<sup>16</sup> This is due to the fact that the Einstein equations, relating curvature and energy-momentum content of all fields other than gravitational fields, as already mentioned in the above do not need to be imposed in the analogy.

It is important to recognize that the form (2) of the wave equation is valid generally (with a possible additional scalar potential term). That is, a generally covariant curved space-time wave equation can be formulated not just for the velocity perturbation (sound) potential in an irrotational Euler fluid, for which we have introduced the effective metric concept. If the spectrum of excitation (in the local rest frame) is linear,  $\omega = c_{\text{prop}}k$ , where  $c_{\text{prop}}$  is the propagation speed of *some* collective excitation, the statement that an effective space-time metric exists is true, provided we only consider wave perturbations of a single scalar field which constitutes the fixed classical background. The argument to reach this conclusion is as follows.

Given that the action density  $\mathcal{L}$  is a functional of  $\phi$  and its space-time derivatives  $\partial_\mu\phi$ , i.e.  $\mathcal{L} = \mathcal{L}[\phi, \partial_\mu\phi]$ , we expand the action to quadratic order in the fluctuations around some stationary classical background solution  $\phi_0$  of the Euler-Lagrange equations. For any Lagrangian of the specified form, the wave equation for pertur-

<sup>b</sup>This fact implies that metrics with nonvanishing spatial curvature, like the Kerr metric, are not amenable within this simple effective metric scheme, which starts from Euler and continuity equations; see for a discussion Ref.<sup>11</sup>.

bations  $\delta\phi \equiv \Phi$  above the background  $\phi_0$  is<sup>17</sup>

$$\partial_\mu \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \partial(\partial_\mu \phi)} \Big|_{\phi=\phi_0} \partial_\nu \Phi \right) - \left( \frac{\partial^2 \mathcal{L}}{\partial\phi \partial\phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \partial\phi} \right\} \right) \Big|_{\phi=\phi_0} \Phi = 0. \quad (6)$$

The above equation in covariant notation reads

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) - V_{\text{eff}}(\phi_0) \Phi = 0, \quad (7)$$

where  $V_{\text{eff}}(\phi_0)$  is a background-field dependent effective potential (equal to the second term in round brackets in equation (6) divided by  $\sqrt{-g}$ ). The potential  $V_{\text{eff}}$  may, for example, in the simplest case contain an effective mass of the scalar field, such that the wave equation becomes Klein-Gordon like,  $\square\Phi = -m^2 c_{\text{prop}}^4 \Phi$ .

The effective metric coefficients are, up to an (again dimension dependent) conformal factor given by:

$$g_{\mu\nu}(\phi_0) \propto \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \partial(\partial_\mu \phi)} \Big|_{\phi=\phi_0}. \quad (8)$$

The concept of an effective space-time metric therefore applies to every system having a single scalar wave equation of second order in both space and time derivatives, corresponding to perturbations propagating on a fixed classical background, where this background itself determines the metric coefficients.

## 2. The effective metric in Bose-Einstein condensates

We assumed in Eq. (5) that the compressibility  $1/g$  is a constant. This entails that the (barotropic) equation of state reads  $p = \frac{1}{2}g\rho^2$ . We then have, in terms of the interaction between the particles (atoms) constituting the fluid, a contact interaction (pseudo-)potential,  $V(\mathbf{x} - \mathbf{x}') = g\delta(\mathbf{x} - \mathbf{x}')$ . This is indeed the case for the *dilute* atomic gases forming a Bose-Einstein condensate. Well below the transition temperature, they are described to good accuracy by the Gross-Pitaevskiĭ mean field equation for the order parameter  $\Psi \equiv \langle \hat{\Psi} \rangle$ ,<sup>c</sup> representing the expectation value of the quantum field operator  $\hat{\Psi}$ :

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{x}, t) = \left[ -\frac{\hbar^2}{2m} \Delta + V_{\text{trap}}(\mathbf{x}, t) + g|\Psi(\mathbf{x}, t)|^2 \right] \Psi(\mathbf{x}, t). \quad (9)$$

The Madelung transformation reads  $\Psi = \sqrt{\rho} \exp[i\phi]$ , where  $\rho$  yields the condensate density and  $\phi$  is the velocity potential; it allows for an interpretation of quantum theory in terms of hydrodynamics.<sup>19</sup> Namely, identifying real and imaginary parts on left- and right-hand sides of (9), respectively, gives us the two equations

$$-\hbar \frac{\partial}{\partial t} \phi = \frac{1}{2} m \mathbf{v}^2 + V_{\text{trap}} + g\rho - \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \equiv \mu + p_Q, \quad (10)$$

<sup>c</sup>Observe that  $\langle \hat{\Psi} \rangle \neq 0$  breaks particle number conservation (the global U(1) invariance); for a review of the consequences see Ref.<sup>18</sup>.

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (11)$$

The first of these equations is the Josephson equation for the superfluid phase, which corresponds to the Bernoulli equation of classical hydrodynamics, where the usual velocity potential of irrotational hydrodynamics equals the superfluid phase  $\phi$  times  $\hbar/m$ , such that  $\mathbf{v} = \hbar \nabla \phi / m$ . The latter equation implies that the flow is irrotational save on singular lines, around which the *wave function phase*  $\phi$  is defined only modulo  $2\pi$ . Therefore, circulation is quantized,<sup>20</sup> and these singular lines are the center lines of quantized vortices. The isothermal chemical potential  $\mu$  (which we have chosen to incorporate the kinetic energy term  $\frac{1}{2}m\mathbf{v}^2$ ), is augmented by the “quantum pressure term”  $p_Q \equiv -\frac{\hbar^2}{2m}(\Delta\sqrt{\rho})/\sqrt{\rho}$ , which is the one genuine quantum term in (10), because  $p_Q \propto \hbar^2$  (observe that the first order in  $\hbar$  may be incorporated into the velocity potential). The second equation (11) is the continuity equation for conservation of particle number, i.e., atom number in the superfluid gas. The dynamics of the weakly interacting, dilute ensemble of atoms is thus that of a perfect Euler fluid with quantized circulation of singular vortex lines, which are the only vortical excitations in a superfluid. The fact that it is an Euler fluid is true save for regions in which the density rapidly varies and the quantum pressure term  $p_Q$  becomes relevant, which happens on scales of order the coherence length  $\xi_0 = \hbar/\sqrt{2gm\rho_0}$ , where  $\rho_0$  is a constant asymptotic density far away from the density-depleted (or possibly density-enhanced) region. The quantum pressure becomes relevant in the depleted-density cores of quantized vortices, or at the low-density boundaries of the system, and is negligible outside these domains of rapidly varying or low density.

The whole armoury of space-time metric description of excitations, explained in the last section, which is based on the Euler and continuity equations, is then valid for phonon-like excitations of a Bose-Einstein condensate, with the space-time metric (4), as long as we are outside the core of quantized vortices,<sup>d</sup> and far enough from the boundaries of the condensate.

### 3. Nonuniqueness of the quasiparticle content of a Bose-Einstein condensate

Quasiparticles are the fundamental entities used to describe an interacting condensed matter system in a particle picture, that is, in a suitable Fock space. On the microscopic level, if the elementary constituents interact by two-body forces, we are given a second quantized Hamiltonian operator of the form  $\hat{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + V_{\mathbf{k}\mathbf{k}'} \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'}$ , where  $V_{\mathbf{k}\mathbf{k}'}$  are the matrix elements for two-particle interaction in a plane wave basis, and  $\epsilon_{\mathbf{k}}$  are the bare single particle energies of the “elementary” bosons or fermions, which are created by the operators  $\hat{a}_{\mathbf{k}}^\dagger$  from

<sup>d</sup>A treatment of sound wave propagation in the presence of vorticity, and the corrections to Eq. (2) arising therefrom, may be found in Ref. <sup>21</sup>.

the proper particle vacuum. One then employs a unitary, i.e. operator-algebra-conserving Bogoliubov transformation<sup>22</sup> to another set of quasiparticle operators  $\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}}^\dagger$ , see Eq. (13) below. This gives the Hamiltonian the reduced diagonal form  $\hat{H}_{\text{red}} = \sum_{\mathbf{k}} \omega(\mathbf{k}) \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} + O(\hat{b}^3)$ , where  $O(\hat{b}^3)$  represents additional terms of higher order than quadratic, which are supposed to be small compared to the leading diagonalized part of the Hamiltonian, for the picture of noninteracting quasiparticles to make sense. These quasiparticles possess a (possibly spatially anisotropic) dispersion relation  $\omega = \omega(\mathbf{k})$  which is linear for various important classes of quasiparticles,  $\omega(k) \propto k$ . Among these classes of collective excitations are, e.g., phonons, antiferromagnetic magnons,<sup>23</sup> or the excitations around the gap nodes of the  $p$ -wave superfluid  $^3\text{He-A}$ .<sup>24</sup>

Phonons are the small momentum quanta of the sound field in solids or fluids, with  $\omega(k) = c_s k$  for a medium at rest. Their classical equation of motion in a perfect fluid is the generally covariant wave equation (2), with the effective space-time metric (4). In the simplest case, for a homogeneous medium with space and time independent density, and a constant speed of sound  $c_s$ , the quantized velocity potential of the phonons reads<sup>25</sup>

$$\hat{\Phi}(\mathbf{x}, t) = \sum_{\mathbf{k}} \sqrt{\frac{\hbar c_s}{2V \rho_0 k}} \left[ \hat{b}_{\mathbf{k}} e^{-ic_s k t + i\mathbf{k} \cdot \mathbf{x}} + \hat{b}_{\mathbf{k}}^\dagger e^{ic_s k t - i\mathbf{k} \cdot \mathbf{x}} \right], \quad (12)$$

where  $V$  is the quantization volume of the system, and  $\rho_0$  the constant background density. That is, the quantum excitation field in a homogeneous medium may generally be decomposed into plane waves, with the appropriate frequencies as a function of momentum stemming from the dispersion relation, here  $\omega = c_s k$ . Note, in particular, that for this spatially and temporally homogeneous case, the statement that we observe positive frequency (energy) with respect to the laboratory time interval  $dt$  is unique, that is, it can be made independent of time and space. In an inhomogeneous fluid, this is (generally) no longer the case, and the notion of an excitation having positive energy may depend on where the detector is located in the fluid, if it is at rest relative to the fluid or moves, and what its natural time interval is. The latter may be different from that of the laboratory, due to the particular way the detector couples to the fluid, see section 4.

### 3.1. Operator basis dependence of quasiparticle content

The number of particles assigned to the quantum field  $\hat{\Phi}$  is unique with respect to a certain given state  $|\Theta\rangle$  of the quantum field,  $n_{\mathbf{k}}(\Theta) = \langle \Theta | \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} | \Theta \rangle$ , provided we decompose  $\hat{\Phi}$  into modes associated to the operators  $\hat{a}_{\mathbf{k}}$  and their Hermitian conjugates. The number of particles is, in particular, zero for the *vacuum state with respect to the field operator  $\hat{a}_{\mathbf{k}}$* , defined by  $\hat{a}_{\mathbf{k}}|0\rangle = 0$ . However, it need not be zero with respect to another set of quasiparticle operators  $\hat{b}_{\mathbf{k}}$ , which has a *different* vacuum  $|\bar{0}\rangle$ .

To demonstrate the basis dependence of quasiparticle number, we use a general

Bogoliubov transformation for bosons, the class of (quasi-)particles we consider, of the general form

$$\hat{a}_i = \sum_{\mathbf{k}} \alpha_{i\mathbf{k}} \hat{b}_{\mathbf{k}} + \beta_{i\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger, \quad (13)$$

where  $\mathbf{k}$  represents a set of quantum numbers, not necessarily the plane waves used in Eq. (12). The coefficients in the Bogoliubov transformation must fulfill certain conditions for the transformation to be unitary, i.e., to preserve the bosonic commutation relations for the new operators,  $[\hat{b}_i, \hat{b}_{\mathbf{k}}] = \delta_{i\mathbf{k}}$ ,  $[\hat{b}_i, \hat{b}_{\mathbf{k}}] = 0$ ,  $[\hat{b}_i^\dagger, \hat{b}_{\mathbf{k}}^\dagger] = 0$ . As a consequence of this defining unitary character, the following conditions on the transformation coefficients must hold:

$$\sum_{\mathbf{k}} \alpha_{i\mathbf{k}} \alpha_{j\mathbf{k}}^* - \beta_{i\mathbf{k}} \beta_{j\mathbf{k}}^* = \delta_{ij}, \quad \sum_{\mathbf{k}} \alpha_{i\mathbf{k}} \beta_{j\mathbf{k}} - \beta_{i\mathbf{k}} \alpha_{j\mathbf{k}} = 0. \quad (14)$$

By using the transformation (13), it is straightforwardly shown that the number of  $\hat{a}_{\mathbf{k}}$  particles in  $|\bar{0}\rangle$  is given by  $\langle \bar{0} | \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} | \bar{0} \rangle = \sum_{\mathbf{k}'} |\beta_{\mathbf{k}'\mathbf{k}}|^2$ . The old operator  $\hat{a}_{\mathbf{k}}$  does not annihilate the new vacuum  $|\bar{0}\rangle$  (and vice versa), and what looks empty in one quasiparticle vacuum may be full of quasiparticles in another. Related to this (formal) operator-basis dependence is the fact that the actually *detected* number of particles is strongly *observer* dependent, as opposed to the formally defined quantity  $n_{\mathbf{k}}(\Theta)$ , which refers to one particular quasiparticle state  $|\Theta\rangle$ . The detected number of particles depends, in particular, on how the detector actually couples to the field  $\hat{\Phi}$  whose quanta it measures. Various couplings of the detector, for example to different powers of the fluid density, will influence the quasiparticle basis in which the detector measures, and thus the quasiparticle number detected.

Now, the salient point is that because the phonon field  $\hat{\Phi}$  is a *relativistic* quantum field, we will be able to map the observer dependence just described, which is general and holds for any sort of quasiparticles, to the observer dependence experienced by proper relativistic quantum fields in curved or flat space-time.<sup>26,28</sup> The observer dependence is of kinematical origin, i.e., it originates in the fact that the relativistic quantum field propagates in a space-time of Lorentzian signature, cf. the discussion of the Hawking radiation analogue phenomenon in Ref.<sup>29</sup>, and is therefore fully within the capabilities of our proposed analogy.

### 3.2. Scaling ansatz in expanding Bose-Einstein condensates

To model the quasiparticle analogue of expanding universes, we will make use of expanding Bose-Einstein condensates, which are produced by reducing the trapping potential strength, i.e. the harmonic trapping frequency with time, or by increasing the interaction coupling constant. Firstly, it is thus appropriate to describe the evolution of density and velocity distribution in the expanding gas by discussing the so-called scaling procedure established in Refs.<sup>33,34</sup>. The scaling procedure introduces a set of generally three scaling variables,  $b_i = b_i(t)$ , which are a function of time only. Using these scaling variables, one writes for the (Cartesian) co-ordinate



vector components  $x_{bi} = x_i/b_i$ ; for the scaled co-ordinate vector we use the short-hand notation  $\mathbf{x}_b \equiv \sum_i \mathbf{e}_i x_i/b_i$ . It may then be shown that the evolution of the Bose-Einstein-condensed gas cloud is described, starting from the initial density and velocity potential distributions  $\rho = \rho_{\text{init}}(\mathbf{x}, t=0)$ ,  $\phi = \phi_{\text{init}}(\mathbf{x}, t=0)$ , by the following density and velocity distributions (from here on we take  $\hbar = m = 1$ ):<sup>33,34</sup>

$$\rho(\mathbf{x}, t) \Rightarrow \frac{\tilde{\rho}_{\text{init}}(\mathbf{x}_b)}{\mathcal{V}}, \quad (15)$$

$$\mathbf{v} = \nabla \phi(\mathbf{x}, t) \Rightarrow \mathbf{v} = \nabla \phi = \sum_i \frac{\dot{b}_i}{b_i} x_i + \nabla \tilde{\phi}(\mathbf{x}_b, t). \quad (16)$$

This is true provided the scaling parameters  $b_i$  fulfill the equations<sup>35,39</sup>

$$\ddot{b}_i + \omega_i^2(t) b_i = \frac{g(t)}{g(0)} \frac{\omega_{0i}^2}{\mathcal{V} b_i}, \quad (17)$$

where  $\omega_{0i}$  are the initial trapping frequencies and the dimensionless “scaling volume” reads  $\mathcal{V} = \prod_i b_i$ . The dots are time derivatives with respect to laboratory time  $t$  here and in what follows.<sup>e</sup> We have taken into account in the above equation that the particle interaction can be varied in lab time,  $g = g(t)$ , by means of a suitable Feshbach resonance;<sup>36,37</sup>  $g(0)$  is the initial coupling constant. We designate “scaling basis” quantities with a tilde. For example, the (stationary) initial density distribution  $\rho_{\text{init}}(\mathbf{x})$  gives us  $\tilde{\rho}_{\text{init}}(\mathbf{x}_b)$  if we replace  $\mathbf{x} \rightarrow \mathbf{x}_b$ . The scaling evolution is exact in a Thomas-Fermi approximation which neglects the quantum pressure term  $p_Q$  and the kinetic energy  $\frac{1}{2} m \mathbf{v}^2$ . Since Bose-Einstein condensates were experimentally created, the scaling solution has routinely been employed to interpret the time-of-flight pictures with which they are visualized as well as analyzed.<sup>2</sup>

We define a “scaling time” variable by

$$\frac{d\tau_s}{dt} = \frac{g(t)/g(0)}{\mathcal{V}}, \quad (18)$$

and the  $\tau_s$  dependent *scaling functions*  $F_i$  by

$$F_i(\tau_s) = \frac{\mathcal{V}}{b_i^2} \frac{g(0)}{g(\tau_s)} = \frac{1}{b_i^2} \frac{dt}{d\tau_s}. \quad (19)$$

In terms of these quantities, the effective second order action for the scaling basis fluctuations of the phase of the superfluid order parameter  $\delta\tilde{\phi}(\mathbf{x}_b, \tau_s) \equiv \Phi(\mathbf{x}_b, \tau_s)$  takes on the particularly simple diagonal form<sup>39</sup>

$$\bar{S}^{(2)} = \int d\tau_s d^3x_b \frac{1}{2g(0)} \left[ - \left( \frac{\partial}{\partial \tau_s} \delta\tilde{\phi} \right)^2 + \tilde{c}^2 F_i (\nabla_{bi} \delta\tilde{\phi})^2 \right], \quad (20)$$

where the scaling speed of sound  $\tilde{c} = \sqrt{g(0)\tilde{\rho}_{\text{init}}(\mathbf{x}_b)}$ . Because of the fact that this action does not mix spatial and temporal derivatives, the resulting line element

<sup>e</sup>Note that by special convention we generally do not sum over latin indices  $i, j, \dots$ , but only over greek indices  $\mu, \nu, \dots$ .

in the scaling variables, according to the identification displayed in (5) is diagonal (does not possess  $g_{0i}$  terms), and reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{\tilde{c}}{g(0)} \sqrt{F_x F_y F_z} [-\tilde{c}^2 d\tau_s^2 + F_i^{-1} dx_{bi}^2]. \quad (21)$$

We now consider for simplicity the isotropic case,  $b_i \equiv b$ , which implies

$$F_i \equiv F = b^{D-2} \frac{g(0)}{g}. \quad (22)$$

The generalization of the mode expansion in Eq. (12) to inhomogeneous expanding Bose-Einstein condensates then takes, in this isotropic case, the form<sup>39</sup>

$$\hat{\Phi}(\mathbf{x}_b, \tau_s) = \sum_n \sqrt{\frac{g(0)}{2\tilde{V}\epsilon_n}} \phi_n(\mathbf{x}_b) [\hat{b}_n \chi_n(\tau_s) + \hat{b}_n^\dagger \chi_n^*(\tau_s)]. \quad (23)$$

The functions  $\phi_n(\mathbf{x}_b)$  are the stationary solutions of the Gross-Pitaevskii equations for excitations above the initial ground state, designated by the (set of) quantum numbers  $n$  with initial energies  $\epsilon_n$ . The initial Thomas-Fermi quantization volume is  $\tilde{V}$ ; in the hard-walled cubic box limit the modes are plane waves,  $\phi_n(\mathbf{x}_b) \rightarrow \exp[i\mathbf{k} \cdot \mathbf{x}_b]$ .

The temporal mode functions  $\chi_n(\tau_s)$  satisfy the second order ordinary differential equation<sup>39</sup>

$$\frac{d^2}{d\tau_s^2} \chi_n + F(\tau_s) \epsilon_n^2 \chi_n = 0. \quad (24)$$

The case when  $F$  is a constant (unity) is particular. In this case, the quantum state of the quasiparticle excitations remains unchanged for increasing  $\tau_s$ , and a given initial quasiparticle vacuum, *in the scaling basis* with the associated quasiparticle operators defined by (23), remains empty forever. That is, no quasiparticles are created in that basis, although the superfluid may be in a highly nonstationary motional state, obtained by changing the trapping  $\omega = \omega(t)$  rapidly with time. However, a detector which measures in a quasiparticle basis different from the scaling basis, for example due to its particular coupling to the superfluid, may still detect that quasiparticles are “created”. We will come back to this possibility in section 4 below, when we discuss the purely choice-of-observer related phenomenon of a thermal state in a quasiparticle basis belonging to one particular space-time, the de Sitter space-time of cosmology.

### 3.3. “Cosmological” quasiparticle production

Consider now the general case that the scaling function  $F(\tau_s)$  is a function of scaling time  $\tau_s$ . The fact that  $F$  depends on time implies that the statement “the excitation is of positive frequency” (a particle) or of “negative frequency” (antiparticle) for a given propagating wave cannot be held up for all times  $\tau_s$ . This *frequency*

*mixing* implies that quasiparticles are created from the quasiparticle vacuum, because an initially empty scaling basis vacuum state does not remain empty during the evolution of the system, i.e. initially  $\hat{b}_n|0(\tau_s = 0)\rangle = 0$ , but at a later stage  $\hat{b}_n|0(\tau_s)\rangle \neq 0$ .

The fact that annihilation and creation operator parts of the initial vacuum are mixed, as a consequence of (13), is physically due to the fact that quasiparticles are scattered within the course of time at (time dependent) effective potentials. Physically, the  $\tau_s$  dependence of  $F$  furnishes such an effective potential, at which excitations are scattered from negative to positive frequency and vice versa: The equation (24) is formally equivalent to scattering of a non-relativistic particle with energy  $\epsilon_n$  by a potential in  $\tau_s$  space,

$$V(\tau_s) = \epsilon_n^2(1 - F(\tau_s)). \quad (25)$$

At large  $\tau_s$ , the WKB scattering solution of (24) therefore reads:<sup>39</sup>

$$\chi_n = \frac{1}{F^{1/4}} \left( \alpha_n \exp \left[ -i\epsilon_n \int_{-\infty}^{\tau_s} d\tau'_s \sqrt{F(\tau'_s)} \right] + \beta_n \exp \left[ i\epsilon_n \int_{-\infty}^{\tau_s} d\tau'_s \sqrt{F(\tau'_s)} \right] \right), \quad (26)$$

where the scattering amplitudes are related via the particle flux conservation condition  $|\alpha_n|^2 - |\beta_n|^2 = 1$ . The quantity  $N_n = |\beta_n|^2$  can be interpreted as the number of *scaling basis* quasiparticles created from the initially empty scaling vacuum, due to the time dependent scattering of excitations moving in the nonstationary condensate.

In the WKB approximation, the amplitudes  $\alpha_n$  and  $\beta_n$  are connected in a simple way:

$$\beta_n = \exp \left[ -\frac{\epsilon_n}{2T_0} \right] \alpha_n, \quad (27)$$

where the inverse temperature  $1/T_0$  is given by the integral

$$\frac{1}{T_0} = \Im \left[ \int_{\mathcal{C}} \sqrt{F} d\tau_s \right], \quad (28)$$

and  $\mathcal{C}$  is the contour in the complex  $\tau_s$ -plane enclosing the closest to the real axis singular point of the function  $\sqrt{F(\tau_s)}$ .<sup>40</sup> This gives us the number of particles created in the mode  $n$ , using the flux conservation condition  $|\alpha_n|^2 - |\beta_n|^2 = 1$ ,

$$N_n = |\beta_n|^2 = \frac{1}{\exp[\epsilon_n/T_0] - 1}. \quad (29)$$

The distribution of the created quasiparticles follows a thermal bosonic distribution (Planck spectrum), at a temperature  $T_0$ . The adiabatic evolution of trapped gases hence leads to “cosmological” quasiparticle creation with thermal occupation numbers in the scaling basis. The temperature  $T_0$  occurring in the Planck distribution above depends on the details of the scaling evolution, i.e., on the specific superfluid dynamics imposed by the solution of Eqs. (17), that is,  $T_0$  is a functional of the temporal evolution  $\omega_i = \omega_i(t)$ .<sup>38,39</sup>

We use the term “cosmological” in context with the plain condensed-matter fact that quasiparticles are created in the scaling basis. We now justify this by comparing our effective Bose-Einstein condensate metric (21) to line elements which constitute cosmological solutions of the Einstein equations. For example, let  $d\tau_s = dt$  by properly adjusting  $g = g(t)$ , thus choosing the coupling constant’s time evolution to be given by  $g(t) = g(0)\mathcal{V}(t)$ , cf. Eq. (18). We then obtain that (21) equals (up to the conformal factor  $\tilde{c}/(g(0)\mathcal{V})$ ) an anisotropic version of the spatially flat Friedmann-Robertson-Walker (FRW) universe:<sup>41</sup>

$$ds^2 = \frac{\tilde{c}}{g(0)\mathcal{V}} \left[ -\tilde{c}^2 dt^2 + \sum_i b_i^2 dx_{bi}^2 \right]. \quad (30)$$

In the standard spatially isotropic form of the FRW metric, all  $b_i = b$  are equal.

To obtain the exact equivalence to a spatially flat FRW metric, we have to assume in addition that  $\tilde{c}$  is spatially independent, which is fulfilled close to the center of the gas cloud, where the parabolic density profile is approximately flat, and  $\tilde{c}$  is essentially a constant. In the spatially isotropic case,

$$H = \frac{\dot{b}}{b} \quad (31)$$

is the Hubble “constant”, which obviously is a constant only if the gas expands exponentially in laboratory time,  $b \propto \exp[Ht]$ , just as we need exponentially rapid expansion for a constant Hubble parameter in inflationary cosmological models.<sup>42,43</sup>

We therefore come to the remarkable conclusion that the co-ordinate scaling factor  $b$  of the Bose-Einstein condensate quasiparticle universe, occurring in the equations of motion of a nonrelativistic condensed matter system, has exactly the same meaning, in the quasiparticle world, as the cosmological scale factor of the Universe proper.

#### 4. Gibbons-Hawking effect in de Sitter space-time

We now treat the case that the scaling factors  $F_i$  in (19) are constants, i.e., do not depend on scaling time. Therefore, following Eqs. (23) and (24), no scaling basis quasiparticles are created through negative and positive frequency mixing. The superfluid can still be in highly nonstationary motion, though: The time evolution is according to (19) prescribed by

$$b_i^2(t) = C_i \frac{g(0)}{g(t)} \mathcal{V}(t) \iff \frac{b_i}{\prod_{k \neq i} b_k} = C_i \frac{g(0)}{g(t)}, \quad (32)$$

cf. Eq. (19), where the  $C_i$  are constants. However, although the superfluid is in motion, no dissipation through intrinsic quasiparticle creation takes place, because there exists the Fock space “scaling” basis, in which no quasiparticles are created from the scaling vacuum, and the energy of that particular superfluid vacuum is

conserved. A particular instance is the isotropic 2D case,  $b_1 = b_2 = b$ , where for constant  $g(t) = g(0)$  the condition on  $F$  being constant is fulfilled.<sup>f</sup>

In de Sitter space-time space is empty and flat (that is, the constant time slices are Euclidean space), and all of the curvature of space-time is encoded in a nonvanishing cosmological constant  $\Lambda$ . For obvious reasons, the de Sitter space-time is very popular in the quantum field theoretical treatment of cosmological theories,<sup>4,27</sup> because it highlights the crucial cosmological role the energy density of all conceivable quantum fields taken together might play: The vacuum energy density may constitute the dominant effective source term for space-time curvature in the Einstein equations.

The Gibbons-Hawking effect for geodesic observers<sup>28</sup> in such a de Sitter space-time is the curved space-time analogue of the Unruh-Davies effect.<sup>4,26</sup> The latter consists in the fact that a constantly accelerated detector moving in the flat (purportedly “empty”) space-time vacuum, responds as if it were placed in a thermal bath of (quasi-)particles with temperature proportional to its acceleration. Observer-related phenomena are at the heart of quantum field theory on nontrivial, and generally curved, space-time backgrounds. They tell us, in particular, that the particle content of a given quantum field depends on the (motional) state of the detection apparatus, which is verifying that there are particles by its “clicks”. More technically speaking, the dependence of the particle content of quantum fields in curved space-time is rooted in the non-uniqueness of canonical field quantization in Riemannian spaces.<sup>30</sup> It is of fundamental importance to make these observer-dependent effects measurable, because such a measurement constitutes, *inter alia*, a consistency check for an all-important concept of standard quantum field theory, namely that the quantization of a given field is carried out on a *fixed* space-time background.

That the experimental verification of the observer dependence is exceedingly difficult with light becomes readily apparent if we calculate the Unruh-Davies temperature: The result is that it equals  $T_{\text{Unruh}} = [\hbar/(2\pi k_{\text{B}} c_{\text{L}})]a = 4 \text{ K} \times a[10^{20} g_{\oplus}]$ , where  $a$  is the acceleration of the detector in Minkowski space ( $g_{\oplus}$  is the gravity acceleration on the surface of the Earth), and  $c_{\text{L}}$  the speed of light. The huge accelerations needed to obtain measurable values of  $T_{\text{Unruh}}$  make it obvious that an observation of the effect with light (photons) is decidedly a less than trivial undertaking. Although proposals for a measurement with ultraintense short pulses of electromagnetic radiation have been put forward in, e.g., Refs.<sup>31,32</sup>, it is less than obvious how the thermal spectra associated to the effect, which still furnish tiny contributions to the total energy, should be discernible from the background dominated by the ultraintense lasers used to create large accelerations of the elementary particles contained in the plasma.

<sup>f</sup>Cf. Ref.<sup>44</sup>, where the fact of superfluid vacuum energy conservation is explained from a different (SO(2,1) symmetry) perspective, and Ref.<sup>45</sup> for a quality factor measurement of breathing (monopole) oscillations in a cylindrical geometry.

#### 4.1. An isotropic de Sitter universe in a harmonic trap

We first discuss the simplest case of an isotropically expanding gas in a harmonic trap. We will see that this case also serves a pedagogical purpose, because it forces us to distinguish between purely observer-related phenomena with thermal spectrum, which by definition all have  $F_i(\tau_s) \equiv 1$ , and “cosmological” particle creation with a thermal spectrum, as defined in section 3.3, for which the scaling functions  $F_i(\tau_s)$  in (19) depend explicitly on scaling time.

One can create de Sitter universes in an expanding gas by letting it expand exponentially,  $b(t) \propto \exp[Ht]$  where  $H$  is the Hubble constant of cosmological expansion, equalling  $H = \Lambda/D$  for the de Sitter universe discussed, where  $D$  is the spatial dimension. The de Sitter metric in its standard form, using as its time co-ordinate the “cosmological” time  $\tau_c$ , reads<sup>41</sup>

$$ds^2 = \frac{c_0}{g(0)\mathcal{V}} [-c_0^2 d\tau_c^2 + e^{2H\tau_c} d\mathbf{x}_b^2], \quad (33)$$

where  $c_0 \equiv \tilde{c}(\mathbf{0})$  is the central scaling (i.e., initial) speed of sound. In the present case of exponential expansion, the “cosmological” time interval occurring in the metric above is equal to both the laboratory and the scaling time interval,  $d\tau_c = d\tau_s = dt$ ; cf. Eq. (30) with  $b_i = b = \exp[H\tau_c]$ .

However, using the isotropic harmonic expansion setup, one has to face the difficulty that in order to give the FRW metric (30) near the center of the trap (i.e., close to  $\mathbf{x}_b = 0$ ) the de Sitter form, one has to increase exponentially the interaction with laboratory time: Because of (31),  $g(t) \propto b^D \propto \exp[DHt] = \exp[c_0\Lambda t]$ . Though the central density also decreases exponentially (like  $b^{-D/2}$ ), the exponential increase of the coupling “constant” incurs strong three-body recombination losses,<sup>46</sup> whose total rate (in the dilute gas case and in three spatial dimensions) is proportional to  $g^4 \rho^2 \propto b^9 = \exp[9c_0\Lambda t]$ . Therefore, within a short time of order  $1/H$ , the Bose-Einstein-condensed gas of interacting single atoms will simply no longer be in existence, because the atoms rapidly form bound states. Such an experiment will leave no time to measure phenomena which depend on the fact that an equilibrium is established; in particular, the thermal equilibrium for the occupation numbers in the de Sitter quasiparticle basis will not be established on such a short time scale.

Even more importantly, though one obtains indeed a thermal spectrum in the de Sitter quasiparticle basis corresponding to the metric (33),  $F(\tau_s) = F(t) = 1/b^2 = \exp[-2Ht]$  depends on time, i.e., it is not a constant in the quasiparticle basis corresponding to the mode functions  $\chi_n \propto \exp[-i\epsilon_n\tau_s]$ . The thermal spectrum obtained therefore is, by our physical definition, the thermal “cosmological” quasiparticle creation discussed in the previous section. It is *not* what we want to observe in our effective de Sitter space-time, namely the purely choice-of-observer related phenomenon Gibbons-Hawking effect, for which no actual quasiparticle “creation” in the scaling basis should take place.

#### 4.2. The 1+1D de Sitter universe in a cigar-shaped cloud

To circumvent the problem that the interaction coupling needs to be increased exponentially with time to obtain a de Sitter universe in 2+1 or 3+1 isotropic space-time dimensions in an isotropic harmonic trap, I and Petr Fedichev have developed the model of a 1+1D de Sitter universe. This 1+1D toy model can be realized in a strongly anisotropic, cigar-shaped Bose-Einstein condensate,<sup>9,10</sup> cf. Fig. 1. In particular, in the proposed experimental setup, no time variation of the coupling constant at all is necessary, which is thus a true “constant” also in time.

The analysis of the excitation modes in a strongly anisotropic, elongated Bose-Einstein condensate is based on the adiabatic separation ansatz<sup>47</sup>

$$\Phi(r, z, t) = \sum_n \phi_n(r) \chi_n(z, t), \quad (34)$$

where  $\phi_n(r)$  is the radial wavefunction characterized by the quantum number  $n$  (only zero angular momentum modes are considered here). The above ansatz incorporates the fact that for strongly elongated traps, i.e., traps for which  $\omega_z \ll \omega_\perp$ , the dynamics of the condensate motion separates into a fast radial motion and a slow axial motion, which are essentially independent. The  $\chi_n(z, t)$  are the mode functions for travelling wave solutions in the  $z$  direction (plane waves for a condensate at rest read  $\chi_n \propto \exp[-i\epsilon_{n,k}t + kz]$ ). The radial motion is assumed to be “stiff” such that the radial part is effectively time independent, because the radial time scale for adjustment of the density distribution after a perturbation is much less than the axial oscillation time scales of interest. The ansatz (34) works independent from the ratio of healing length and radial size of the superfluid cigar. In the limit that the healing length is much less than the radial size, Thomas-Fermi wave functions are used, in the opposite limit, a Gaussian ansatz for the radial part of the wave function  $\phi_n(r)$  is appropriate.

The squared oscillation spectrum of the cigar-shaped condensate cloud reads  $\epsilon_{n,k}^2 = c_0^2 k^2 + 2\omega_\perp^2 n(n+1)$ , where  $c_0 = \sqrt{\mu/2}$ .<sup>47</sup> A de Sitter space-time is then

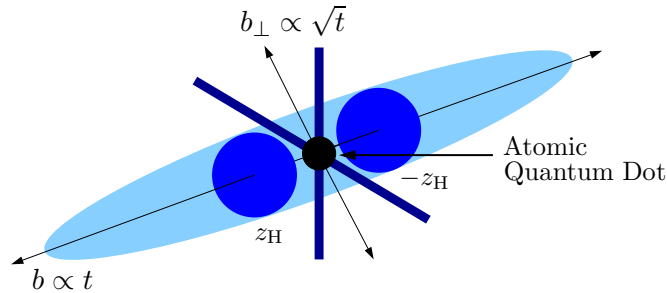


Fig. 1. Expansion of a cigar-shaped Bose-Einstein condensate, with scale factor  $b$  linear in lab time along the axial, and with  $\sqrt{t}$  in the radial direction, scaled by  $b_\perp$ . The stationary horizon surfaces are located at  $\pm z_H$ , respectively. The thick dark lines represent lasers creating an optical potential well in the center of the harmonic trap, which hosts the Atomic Quantum Dot, cf. Fig. 2.

obtained from the effective action for the phase fluctuations of the phonon ( $n = 0$ ) modes near the center of the trap. The corresponding 1+1D action is obtained after integrating out the transverse, strongly confined directions, and reads:<sup>9,10</sup>

$$\begin{aligned} S_0 &= \int dt dz \frac{\pi b_\perp^2 R_\perp^2}{2g} \left[ - \left( \frac{\partial}{\partial t} \chi_0 - v_z \partial_z \right)^2 + \frac{c_0^2}{b_\perp^2 b} (\partial_z \chi_0)^2 \right] \\ &\equiv \frac{1}{2} \int d^{D+1}x \sqrt{-g} g^{\mu\nu} \partial_\mu \chi_0 \partial_\nu \chi_0. \end{aligned} \quad (35)$$

The scaling parameters  $b$  in the axial ( $\omega_z$ ) and  $b_\perp$  in the perpendicular ( $\omega_\perp$ ) directions are functions of time, such that the action fulfills the identification with the action of a scalar field minimally coupled to gravity, as indicated in the second line above, where the metric coefficients are the  $g_{\mu\nu}$  in Eq. (4).

The identification of the above phase-fluctuations action with the action of a scalar minimally coupled to gravity works, that is, the two actions in the first and second lines of (35) are consistent, if we impose the consistency condition that

$$\begin{aligned} \frac{\pi b_\perp^2 R_\perp^2}{g} Z^2 &= \frac{b_\perp \sqrt{b}}{c_0} \\ \Leftrightarrow \quad \frac{b_\perp}{\sqrt{b}} &= 8 \sqrt{\frac{\pi}{2}} \frac{1}{Z^2} \sqrt{\rho_m a_s^3} \left( \frac{\omega_\perp}{\mu} \right)^2 \equiv B = \text{const.}, \end{aligned} \quad (36)$$

where  $Z$  is a renormalization factor according to  $\chi_0 = Z \bar{\chi}_0$ , with  $\bar{\chi}_0$  the renormalized wave function, and  $\rho_m$  the initial central density. The factor  $Z$  does not influence the (classical) equation of motion  $\delta S / \delta \chi_0 = 0$  (it simply drops out), but does influence the response of a detector. We will come back to this point in section 4.3 below, when we discuss the explicit dependence on  $Z$  of the equilibration time scale for the stationary detector state considered, see Eq. (48).

In Refs.<sup>9,10</sup>, we were using an alternative form of the 1+1D de Sitter metric (33). This alternative form is the one used in the original Gibbons-Hawking paper.<sup>28</sup> It reads (we leave out the conformal factor)

$$ds^2 = -c_0^2 (1 - \Lambda z^2) d\tau^2 + (1 - \Lambda z^2)^{-1} dz^2. \quad (37)$$

The time interval  $d\tau$  in the above metric is not the “cosmological” time interval  $d\tau_c$  in the version of the metric displayed in (33). The two metrics may be transformed into each other using the following co-ordinate transformations:

$$\begin{aligned} \exp[-2\sqrt{\Lambda} c_0 \tau] &= \exp[-2\sqrt{\Lambda} c_0 \tau_c] - \Lambda z_b^2, \\ z &= z_b \exp[\Lambda c_0 \tau_c], \end{aligned} \quad (38)$$

giving us  $\tau = \tau(\tau_c, z_b)$ ,  $z = z(\tau_c, z_b)$ , which transforms (37) into (33). The advantage of the form (37) is that it is plain in this form that the de Sitter space-time has an event horizon, located in our 1+1D case at the constant values  $z = z_H = \pm \Lambda^{-1/2}$ .



The quantity  $\Lambda z^2$  in the de Sitter metric (37) must be independent of time. This leads us to the requirement

$$\sqrt{\Lambda}z = \frac{v_z}{c(t)} = \frac{\dot{b}b_\perp}{\sqrt{b}c_0}z = \frac{B\dot{b}}{c_0}z \quad (39)$$

relating the dynamical parameters of our problem to each other ( $c(t)$  is the instantaneous sound velocity at the center of the cloud). These relations imply that  $\dot{b} = \text{const.}$ , and thus that  $b \propto t$ . Then, the cosmological constant  $\Lambda$  becomes independent of time, as is necessary for an analogue de Sitter space-time to be established.

The experimental procedure now is determined to be as follows: Prepare a Thomas-Fermi (i.e., sufficiently large), strongly anisotropically trapped, cigar-shaped condensate. Let it expand in the axial direction linear in lab time by changing the trapping according to the scaling equations (17), such that  $b \propto t$ , and simultaneously expand in the perpendicular direction with the square root of lab time,  $b_\perp \propto \sqrt{t}$ , such that  $B = b_\perp/\sqrt{b}$  in (36) is a constant. Then, a detector “tuned” to the de Sitter space-time (37), i.e., which works in the de Sitter quasiparticle basis, will measure a thermal quasiparticle spectrum, with the de Sitter temperature:<sup>9,10,28</sup>

$$T_{\text{dS}} = \frac{c_0}{2\pi}\sqrt{\Lambda} = \frac{B}{2\pi}\dot{b}. \quad (40)$$

The fact that a thermal spectrum is obtained can be directly derived from the equations of superfluid hydrodynamics, as expounded in Refs.<sup>9,10</sup>. That is, it is not simply postulated due to the (kinematical) analogy with quantum field theory in de Sitter space-time, but is a result of the quantized hydrodynamic equations determining the evolution of the quasiparticle content of the sound field in the de Sitter basis.

The relation between de Sitter time  $\tau$  and the laboratory time is fixed by  $d\tau = dt/(\sqrt{b}\dot{b}_\perp) = dt/(b(t)B) = dt/(\dot{b}tB)$ . [Note that  $d\tau$  and the scaling time interval  $d\tau_s$  defined in Eq. (18) differ;  $d\tau_s = dt/B^2\dot{b}^2 = d\tau/B\dot{b}$ .] The transformation law between  $t$  and the de Sitter time  $\tau$  (on a constant  $z$  detector, such that  $d\tilde{t} = dt$ ), is given by

$$\frac{t}{t_0} = \exp[B\dot{b}\tau], \quad (41)$$

where the unit of lab time  $t_0 \sim \omega_\parallel^{-1}$  is set by the initial conditions for the scaling variables  $b$  and  $b_\perp$ .

It is important to recognize that an effective exponential “acceleration” of the oscillation frequencies, either because of the exponential dependence of laboratory time on scaling time, represented by (41), or coming from the WKB approximation for the scattering amplitudes, and the exponentially small mixing of positive and negative frequency parts resulting therefrom, Eq. (27), is sufficient for the thermal spectrum to be obtained. In particular, though we have discussed a space-time which possesses a horizon, no pre-existing horizons in the parent space-time are necessary *per se* for thermal occupation number distributions to be established, as has been pointed out in Ref.<sup>48</sup>. An explicit example is the Unruh-Davies effect, where this parent, global space-time is simply Minkowski space.

### 4.3. Detecting the thermal de Sitter spectrum

To detect the Gibbons-Hawking effect in de Sitter space-time, one has to set up a detector which measures frequencies in units of the inverse de Sitter time  $\tau$ , rather than in units of the inverse laboratory time  $t$ ; this corresponds to measuring in the proper de Sitter quasiparticle vacuum, where, in particular, positive and negative frequency are defined with respect to  $\tau$ . Only then does one detect quasiparticles which are defined with respect to the de Sitter quasiparticle basis, and refer to a vacuum corresponding to exactly that space-time.

In Refs. <sup>9,10</sup>, I and Petr Fedichev have provided such a detector. We have shown that a “de Sitter basis” detector is realized by an “Atomic Quantum Dot” (AQD) (for a detailed exposition of AQD properties cf. Ref. <sup>49</sup>). The AQD can be implemented in a Bose gas of atoms possessing two hyperfine ground states  $\alpha$  and  $\beta$ ; the level scheme is represented in Fig. 2. The atoms in the state  $\alpha$  represent the expanding Bose-Einstein condensate, and are used to model the expanding de Sitter universe. The AQD itself is formed by trapping atoms in the state  $\beta$  in a tightly confining optical potential  $V_{\text{opt}}$  created by a laser at the center of the cloud. The interaction of atoms in the two internal levels is described by a set of coupling parameters  $g_{cd} = 4\pi a_{cd}$  ( $c, d = \{\alpha, \beta\}$ ), where  $a_{cd}$  are the  $s$ -wave scattering lengths characterizing short-range intra- and inter-species collisions;  $g_{\alpha\alpha} \equiv g$ ,  $a_{\alpha\alpha} \equiv a_s$ , and  $g_{\alpha\beta} \equiv \bar{g}$ . The on-site repulsion between the atoms  $\beta$  in the dot is given by the energy level spacing  $U \sim g_{\beta\beta}/l^3$  between states with a occupation difference of one  $\beta$  atom, where  $l$  is the characteristic size of the ground state wavefunction of atoms  $\beta$  localized in  $V_{\text{opt}}$ . We consider the so-called collisional blockade limit of large  $U > 0$ , where only one atom of type  $\beta$  can be trapped in the dot. This limit assumes that  $U$  is much larger than all other relevant frequency scales in the dynamics of both the AQD and the expanding superfluid, and corresponds to a large “Coulomb blockade gap” in electronic quantum dots.<sup>50</sup> As a result of these assumptions, the collective co-ordinate of the AQD is modeled by a pseudo-spin-1/2 degree of freedom  $\bar{\eta}$ , with spin-up/spin-down state corresponding to occupation of the AQD by a single atom or no atom in the hyperfine state  $\beta$ . A Rabi laser of frequency  $\Omega$ , with a detuning  $\Delta$  from resonance between the two hyperfine levels  $\alpha$  and  $\beta$ , couples atoms of the hyperfine species  $\alpha$ , constituting the expanding cigar-shaped superfluid, into the AQD.

The detector Lagrangian reads (Ref. <sup>10</sup>, cf. the Hamiltonian formulation in Ref. <sup>9</sup>):

$$L_{\text{AQD}} = i \left( \frac{d}{dt} \bar{\eta}^* \right) \bar{\eta} - \Omega \sqrt{\rho_0(0, t) l^3} (\bar{\eta} + \bar{\eta}^*) - \left[ -\Delta + (\bar{g} - g) \rho_0(0, t) + \bar{g} \delta \rho + \frac{d}{dt} \delta \phi \right] \bar{\eta}^* \bar{\eta}. \quad (42)$$

The fact that the detector has the de Sitter basis as its “natural” quasiparticle basis, and therefore measures in de Sitter time, is due to the fact that the term linear

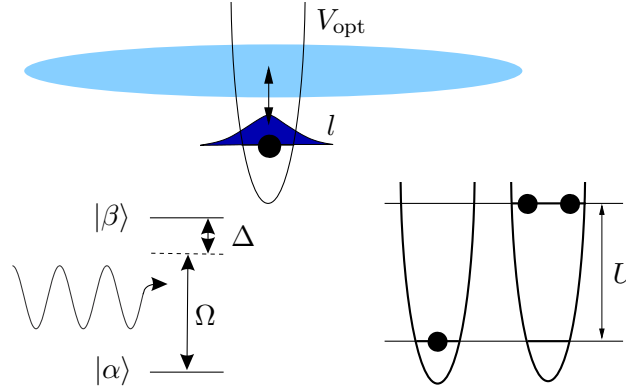


Fig. 2. Level scheme of the “Atomic Quantum Dot”, which is embedded in the superfluid cigar, and created by an optical well for atoms of a hyperfine species different from that of the condensate. Double occupation of the dot is prevented by a collisional blockade mechanism.

in the detector co-ordinate  $\bar{\eta}$  in the Lagrangian couples in a certain manner to the superfluid cigar in which it is embedded. Namely, the laser with frequency  $\Omega$ , causing transitions between the two hyperfine levels  $\alpha$  and  $\beta$ , couples to the *square root* of the central mean-field particle density. This particular coupling (represented by the second term in the first line of the above Lagrangian) is what we need, because  $\sqrt{\rho_0(0,t)} = \sqrt{\rho_m}/bB = \sqrt{\rho_m}/(\dot{b}Bt) = \sqrt{\rho_m}d\tau/dt$ . The fact that the coupling coefficient is proportional to  $d\tau/dt$  is required to establish that the detector can work as a de Sitter detector, because it transforms the detector equations, which are obviously *a priori* in laboratory time, into equations in de Sitter time; see the equations (43) for the temporal evolution of the detector level occupations below.

Adjusting the detuning  $\Delta$  properly, such that  $\Delta(t) = (\bar{g} - g)\rho_0(0,t) = (\bar{g} - g)\rho_m/(\dot{b}^2B^2t^2)$ , the first and the second term in the square brackets in the second line of (42) cancel. One then obtains a simple set of coupled equations for the occupation amplitudes of the state  $\psi = \psi_\beta|\beta\rangle + \psi_\alpha|\alpha\rangle$  of the AQD:

$$i\frac{d\psi_\beta}{d\tau} = \frac{\omega_0}{2}\psi_\alpha + \delta V\psi_\beta, \quad i\frac{d\psi_\alpha}{d\tau} = \frac{\omega_0}{2}\psi_\beta, \quad (43)$$

where  $d\tau$  is the *de Sitter time interval*. Were it not for the density oscillations in the cigar, represented by the potential  $\delta V$ , the above equations (43) would represent a simple two-level system, with frequency splitting  $\omega_0 = 2\Omega\sqrt{\rho_m}l^3$ . The density oscillations contained in the perturbation operator  $\delta V(\tau) = (\bar{g} - g)Bb(\tau)\delta\rho(\tau)$  cause transitions between the two undisturbed *detector eigenstates*  $|\pm\rangle = (|\alpha\rangle \pm |\beta\rangle)/\sqrt{2}$  of the two-level system, which are separated by the energy  $\omega_0$ . The density perturbations in the expanding host superfluid lead to a damping of the Rabi oscillations with frequency  $\omega_0$  between these two states. This constitutes the effect of the de Sitter thermal bath to be observed, where the damping happens on the time scale displayed in (48) below.

The response of the detector, that is, the transition rates between the detector states, can be calculated by evaluating a response function<sup>4</sup> which makes use of the expectation value of the product of two  $b(\tau)\hat{\rho}(\tau)$  operators. The probability per unit time for excitation ( $P_+$ , transition from  $|+\rangle$  to  $|-\rangle$ ) and de-excitation ( $P_-$ , transition from  $|-\rangle$  to  $|+\rangle$ ) of the detector takes the form:<sup>10,26</sup>

$$\begin{aligned}\frac{dP_{\pm}}{d\tau} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int^T \int^T d\tau d\tau' \langle \delta \hat{V}(\tau) \delta \hat{V}(\tau') \rangle e^{\mp i\omega_0(\tau - \tau')} \\ &= \lim_{T \rightarrow \infty} \frac{B^2 (\bar{g} - g)^2}{T} \int^T \int^T d\tau d\tau' \langle b(\tau) \delta \hat{\rho}(\tau) b(\tau') \delta \hat{\rho}(\tau') \rangle e^{\mp i\omega_0(\tau - \tau')}.\end{aligned}\quad (44)$$

The second-quantized solution of the hydrodynamic equations for the density fluctuations above the superfluid ground state in the expanding cigar-shaped Bose-Einstein condensate reads<sup>9,10</sup>

$$\delta \hat{\rho} = \sum_k i \sqrt{\frac{\epsilon_{0k}}{4\pi R_{\perp}^2 R_{\parallel} g}} \frac{\partial}{\partial t} \left( \hat{a}_k \exp \left[ -i \int^t \frac{dt' \epsilon_{0k}}{B b^2} + i k z_b \right] \right) + \text{H.c.} \quad (45)$$

Using this solution, and inserting into (44), we have shown<sup>9,10</sup> that at late times  $\tau$  the transition probabilities per unit de Sitter, i.e., per unit detector time satisfy detailed balance conditions. They correspond to *thermodynamic equilibrium* at the temperature  $T_{\text{dS}}$  displayed in (40):

$$\frac{dP_+/d\tau}{dP_-/d\tau} = \frac{n_B}{1 + n_B}, \quad (46)$$

where the Bose-Planck distribution function takes the form familiar from thermodynamics:

$$n_B = \frac{1}{\exp[\omega_0/T_{\text{dS}}] - 1}. \quad (47)$$

The frequency  $\omega_0 \propto \Omega$  can be varied by changing the undressed Rabi frequency  $\Omega$ , varying the intensity of the Rabi laser. The detector thus has a changeable and therefore *tunable* frequency standard, which can be adjusted to scan the above distribution function for a given de Sitter temperature  $T_{\text{dS}}$ .

From the relation (46), we come to the remarkable conclusion that a properly designed detector can “see” a thermal equilibrium distribution in its quasiparticle basis, though it is embedded in a highly nonstationary system with respect to the laboratory frame. The rapidly expanding Bose-Einstein condensate represents this highly nonstationary system, which hosts the de Sitter quasiparticle detector AQD. I stress here again that (46) is an exact *result* obtained by quantizing hydrodynamic fluctuations in a nonstationary superfluid, and not just concluded from a mere comparison of the phonon dynamics in our expanding superfluid with the quantum field theory of photons in de Sitter space-time.

The equilibration time scale of the detector, and thus the time scale on which the Rabi oscillations between the detector states are damped out, is set by the detector frequency standard (the level spacing)  $\omega_0$ , and by the renormalization factor  $Z$ :<sup>10</sup>

$$\tau_{\text{equil}} = Z^{-2} \omega_0^{-1} \propto (\rho_m a_s^3)^{-1/2} (\mu/\omega_{\perp})^2 \omega_0^{-1}. \quad (48)$$

The renormalization factor  $Z$  contained in (36) determines the equilibration rapidity because it physically expresses the strength of detector-field coupling. It is related to the initial diluteness parameter  $D_p(0) \equiv (\rho_m a_s^3)^{1/2}$  of the Bose-Einstein condensate and to the ratio  $\mu/\omega_\perp$ , which determines inasmuch the system is effectively one-dimensional,  $Z^2 \propto D_p(0)(\omega_\perp/\mu)^2$ . To obtain sufficiently fast equilibration, the condensate thus has to be initially not too dilute as well as close to the quasi-1D régime, for which the transverse harmonic oscillator energy scale is of order the energy per particle,  $\mu \sim \omega_\perp$ . These two conditions have another important implication. The ratio of the instantaneous coherence length  $\xi_c(t) = (8\pi\rho_0(0,t)a_s)^{-1/2} \propto t$  and the location of the horizons  $z = z_H = \pm\Lambda^{-1/2}$ , which are *stationary* in the present setup, has to remain less than unity within the equilibration time scale.<sup>§</sup> If this is not the case, the coherence length, which plays the role of the analogue Planck scale (which is time dependent here), exceeds the length scale of the horizon at equilibration, and the concept of “relativistic” phonons propagating on a fixed curved space-time background with local Lorentz symmetry becomes invalid. The ratio  $\xi_c(t)/z_H$  at the lab equilibration time scale  $t = t_{\text{equil}} = t_0 \exp[2\pi(T_{\text{dS}}/\omega_0)(\mu/\omega_\perp)^2 D_p^{-1}(0)]$  following from the de Sitter equilibration time in Eq. (48), expressed in parameters relevant to the experiment, is given by

$$\frac{\xi_c(t_{\text{equil}})}{z_H} = \frac{\pi t_0 T_{\text{dS}}^2}{\rho_m a_s} \exp \left[ 2\pi \frac{T_{\text{dS}}}{\omega_0} \left( \frac{\mu}{\omega_\perp} \right)^2 \frac{1}{D_p(0)} \right]. \quad (49)$$

We see that this ratio changes exponentially with both the initial diluteness parameter  $D_p(0)$  and the quasi-1D parameter  $\mu/\omega_\perp$ . In most currently realized Bose-Einstein condensates,<sup>2</sup> the diluteness parameter  $D_p \sim 10^{-2}$ . Here, we initially need  $D_p(0) \sim O(1)$  to have the condition  $\xi_c(t_{\text{equil}})/z_H < 1$  fulfilled, assuming a reasonably large value of the de Sitter temperature  $T_{\text{dS}}$ . Though the condensate has to be *initially* quite dense, it is to be stressed that the central density decays like  $t^{-2}$  during expansion. Therefore, the rate of three-body recombination losses quickly decreases during the expansion of the gas, and the initially relatively dense Bose-Einstein condensate, which would rapidly decay if left with a  $D_p$  close to unity, can live sufficiently long, the total rate of three-body losses decreasing like  $\rho_0^2(0,t) \propto t^{-4}$ .

## 5. Summary

The primary statement to be drawn from the present article is that phonons, i.e., low-energy linear-dispersion quasiparticles, moving in a spatially and temporally inhomogeneous Bose-Einstein-condensed superfluid gas, are equivalent to photons, the quanta of the electromagnetic field, moving on geodesics in a given curved space-time. We have explored the classical as well as the quantum aspects of this statement.

<sup>§</sup>I thank C. Zimmermann for a pertinent question during a talk given by me at Tübingen, leading to this observation.

On the classical side, the analogy helps to provide us with a simple general means to study quasiparticle propagation in an inhomogeneous medium in motion. An example of such an application is the gravitational lensing effect exerted by a superfluid vortex.<sup>13</sup> On the quantum field theoretical side, we can access within the analogy phenomena which are extremely difficult if not impossible to access with light. One of the fundamentals of quantum field theory, the fact that the particle content of a quantum field depends on the observer, can thus be experimentally verified for the first time. The basic reason that the phenomena in question are (comparatively) easy to simulate in a condensed matter system is that the energy and temperature scales, under which they occur, relative to the typical energy scales of the system, can be changed at will by the experimentalist in a very controlled manner. More particularly, the temperature of the thermal spectrum of phonons to be measured in the Gibbons-Hawking effect can be made relatively large compared to the axial phonon frequencies and the actual temperature of the gas itself, by expanding the condensate cloud rapidly enough. In the condensed matter analogue, the typical energy of the quanta produced can in principle be even made to approach the relevant “Planck” scale, i.e., the point in the quasiparticle energy spectrum where it begins to deviate from being linear.

Finally, on a more adventurous side, one could conceive of carrying out experiments in “experimental” cosmology, as opposed to the currently existing purely “observational” cosmology. In such an experimental approach to matters cosmological, one would try to reproduce under certain specified and, in particular, well-defined initial conditions large-scale features of the cosmos, in the laboratory setting of nonstationary, inhomogeneous superfluid gases.

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